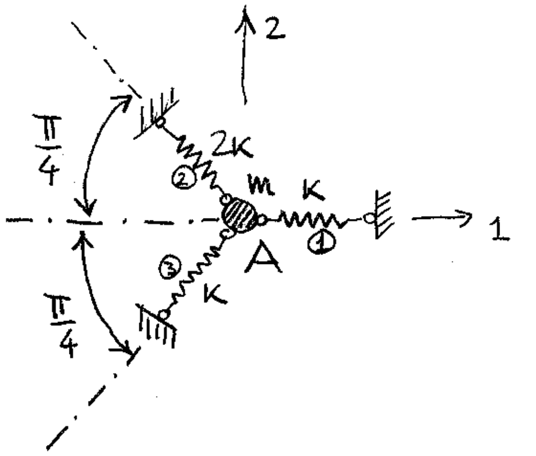
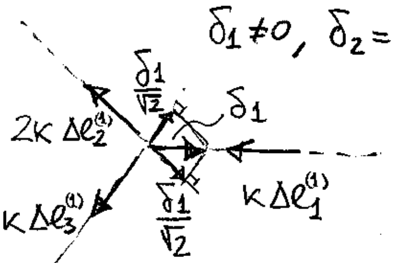
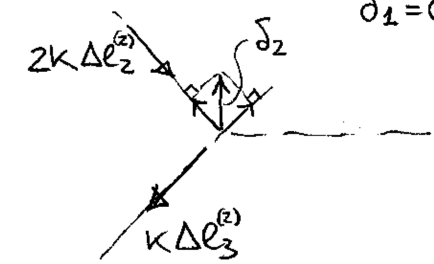


$$\underset{\sim}{s}_A = \delta_1 \underset{\sim}{e}_1 + \delta_2 \underset{\sim}{e}_2$$



$\delta_1 \neq 0,\ \delta_2 = 0$

$$\Delta\ell_1^{(1)} = -\delta_1$$
$$\Delta\ell_2^{(1)} = \delta_1/\sqrt{2}$$
$$\Delta\ell_3^{(1)} = \delta_1/\sqrt{2}$$



$\delta_1 = 0,\ \delta_2 \neq 0$

$$\Delta\ell_1^{(2)} = 0$$
$$\Delta\ell_2^{(2)} = -\delta_2/\sqrt{2}$$
$$\Delta\ell_3^{(2)} = \delta_2/\sqrt{2}$$

Scrittura delle equazioni di equilibrio

Consideriamo separatamente gli spostamenti δ_1, δ_2 per poi utilizzare il principio di sovrapposizione degli effetti.

Gli allungamenti delle molle si ottengono proiettando lo spostamento di A lungo l'asse di ciascuna molla.

L'equilibrio del punto materiale A di massa m si scrive

$$\sum_i \underset{\sim}{f}_i^{(ni)} + \sum_j \underset{\sim}{f}_j^{(in)} = \underset{\sim}{0},$$

cioé la somma delle forze inerziali e delle forze non inerziali è pari a zero.
Nel nostro caso, l'unica forza inerziale vale

$$\underset{\sim}{f}^{in} = -m\ddot{\underset{\sim}{s}}_A = -m\left(\ddot{\delta}_1 \underset{\sim}{e}_1 + \ddot{\delta}_2 \underset{\sim}{e}_2\right)$$

Proiettando tutte le forze agenti lungo la direzione di $\underset{\sim}{e}_1$ si ha:

$$\underbrace{-K\left(\Delta\ell_1^{(1)} + \Delta\ell_1^{(2)}\right)}_{(1^a \text{ molla})} \underbrace{- 2K\left(\Delta\ell_2^{(1)} - \Delta\ell_2^{(2)}\right)\frac{1}{\sqrt{2}}}_{(2^a \text{ molla})} \underbrace{- K\left(\Delta\ell_3^{(1)} + \Delta\ell_3^{(2)}\right)\frac{1}{\sqrt{2}}}_{(3^a \text{ molla})} - m\ddot{\delta}_1 = 0\,;$$

$$-k\delta_1 - 2k\left(\frac{\delta_1}{2} - \frac{\delta_2}{2}\right) - k\left(\frac{\delta_1}{2} + \frac{\delta_2}{2}\right) - m\ddot{\delta}_1 = 0 \;;$$

$$-\frac{5}{2}k\delta_1 + \frac{1}{2}k\delta_2 - m\ddot{\delta}_1 = 0 \,.$$

Analogamente, l'equilibrio lungo la direzione di $\underset{\sim}{e}_2$ si scrive come segue:

$$\underbrace{2k\left(\Delta\ell_2^{(1)} - \Delta\ell_2^{(2)}\right)\frac{1}{\sqrt{2}}}_{(2^a \text{ molla})} - \underbrace{k\left(\Delta\ell_3^{(1)} + \Delta\ell_3^{(2)}\right)\frac{1}{\sqrt{2}}}_{(3^a \text{ molla})} - m\ddot{\delta}_2 = 0 \;;$$

$$2k\left(\frac{\delta_1}{2} - \frac{\delta_2}{2}\right) - k\left(\frac{\delta_1}{2} + \frac{\delta_2}{2}\right) - m\ddot{\delta}_2 = 0 \;;$$

$$\frac{1}{2}k\delta_1 - \frac{3}{2}k\delta_2 - m\ddot{\delta}_2 = 0 \,.$$

Riscrivendo le due equazioni nella forma $\underset{\sim}{M}\ddot{\underset{\sim}{s}}_A + \underset{\sim}{K}\underset{\sim}{s}_A = \underset{\sim}{0}$, con $\underset{\sim}{s}_A \equiv \begin{bmatrix}\delta_1 \\ \delta_2\end{bmatrix}$, otteniamo per le matrici $\underset{\sim}{M}$ e $\underset{\sim}{K}$ le seguenti espressioni:

$$\underset{\sim}{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \,, \quad \underset{\sim}{K} = \begin{bmatrix} \frac{5}{2}k & -\frac{1}{2}k \\ -\frac{1}{2}k & \frac{3}{2}k \end{bmatrix}$$

Per trovare le pulsazioni proprie del sistema, p_1 e p_2, occorre risolvere il seguente problema agli autovalori:

$$\left(\underset{\sim}{K} - p^2\underset{\sim}{M}\right)\bar{\underset{\sim}{s}}_A = \underset{\sim}{0} \,.$$

L'equazione caratteristica si scrive come segue:

$$\det\left(\underset{\sim}{K} - p^2\underset{\sim}{M}\right) = 0 \,.$$

$$\det\begin{bmatrix} \frac{5}{2}k - p^2 m & -\frac{1}{2}k \\ -\frac{1}{2}k & \frac{3}{2}k - p^2 m \end{bmatrix} = 0$$

$$\frac{15}{4}k^2 + p^4 m^2 - 4\,km\,p^2 - \frac{1}{4}k^2 = 0$$

$$m^2 (p^2)^2 - 4km\,p^2 + \frac{14}{4}k^2 = 0 \Rightarrow p^2 = \frac{2km \pm \sqrt{4k^2m^2 - \frac{14}{4}k^2m^2}}{m^2}$$

$$p_1^2 = \frac{k}{m}\left(2 - \frac{1}{\sqrt{2}}\right) \quad , \quad p_2^2 = \frac{k}{m}\left(2 + \frac{1}{\sqrt{2}}\right)$$

Troviamo l'autovettore corrispondente a p_1:

$$\left(\underset{\sim}{K} - p_1^2 \underset{\sim}{M}\right) \underset{\sim}{\bar{S}}_A^{(1)} = 0 \Rightarrow \frac{5}{2}k - \left(2 - \frac{1}{\sqrt{2}}\right)k\ \bar{\delta}_1^{(1)} - \frac{1}{2}k\ \bar{\delta}_2^{(1)} = 0$$

$$k\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)\bar{\delta}_1^{(1)} - \frac{1}{2}k\ \bar{\delta}_2^{(1)} = 0 \quad ; \quad \bar{\delta}_2^{(1)} = \left(1 + \sqrt{2}\right)\bar{\delta}_1^{(1)}$$

$$\bar{S}_A^{(1)} \equiv \begin{bmatrix} 1 \\ 1 + \sqrt{2} \end{bmatrix} \bar{\delta}_1^{(1)} .$$

Il secondo autovettore si ottiene dall'equazione $\left(\underset{\sim}{K} - p_2^2 \underset{\sim}{M}\right)\underset{\sim}{\bar{S}}_A^{(2)} = 0$ o, equivalentemente, trovando il vettore ortogonale a $\underset{\sim}{\bar{S}}_A^{(1)}$:

$$\underset{\sim}{\bar{S}}_A^{(2)} \circ \underset{\sim}{\bar{S}}_A^{(1)} = 0 \Rightarrow \bar{\delta}_1^{(2)}\bar{\delta}_1^{(1)} + \bar{\delta}_2^{(2)}\left[\left(1+\sqrt{2}\right)\bar{\delta}_1^{(1)}\right] = 0$$

$$\bar{\delta}_1^{(2)} = -\left(1+\sqrt{2}\right)\bar{\delta}_2^{(2)} \Rightarrow \bar{S}_A^{(2)} \equiv \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix} \bar{\delta}_2^{(2)}$$

Studiamo ora lo stesso problema utilizzando il metodo energetico.

$$E_{cin} = \frac{1}{2} m \dot{s}_A^2$$

$$E_{el} = \sum_{i=1}^{3} \frac{1}{2} K_i (\Delta \ell_i)^2 \quad \text{con } \Delta \ell_i = \Delta \ell_i^{(1)} + \Delta \ell_i^{(2)} \text{ calcolati in precedenza}$$

$$E_{el} = \frac{1}{2} K\, \delta_1^2 + \frac{1}{2}(2K)\left(\frac{\delta_1}{\sqrt{2}} - \frac{\delta_2}{\sqrt{2}}\right)^2 + \frac{1}{2} K \left(\frac{\delta_1}{\sqrt{2}} + \frac{\delta_2}{\sqrt{2}}\right)$$

Calcoliamo $\dot{E}_{TOT} = \dot{E}_{cin} + \dot{E}_{el}$

$$\dot{E}_{cin} = m\left(\dot{\delta}_1 \ddot{\delta}_1 + \dot{\delta}_2 \ddot{\delta}_2\right) \quad \Longrightarrow \quad M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$\dot{E}_{el} = K\, \delta_1 \dot{\delta}_1 + 2K\left(\frac{\delta_1}{\sqrt{2}} - \frac{\delta_2}{\sqrt{2}}\right)\left(\frac{\dot{\delta}_1}{\sqrt{2}} - \frac{\dot{\delta}_2}{\sqrt{2}}\right) + K\left(\frac{\delta_1}{\sqrt{2}} + \frac{\delta_1}{\sqrt{2}}\right)\left(\frac{\dot{\delta}_1}{\sqrt{2}} + \frac{\dot{\delta}_1}{\sqrt{2}}\right)$$

$$= K\left(\delta_1 \dot{\delta}_1 + \delta_1 \dot{\delta}_1 - \delta_1 \dot{\delta}_2 - \delta_2 \dot{\delta}_1 + \delta_2 \dot{\delta}_2 + \frac{1}{2}\left(\delta_1 \dot{\delta}_1 + \delta_1 \dot{\delta}_2 + \delta_2 \dot{\delta}_1 + \delta_2 \dot{\delta}_2\right)\right) =$$

$$= K\left(\frac{5}{2} \delta_1 \dot{\delta}_1 - \frac{1}{2} \delta_1 \dot{\delta}_2 - \frac{1}{2} \delta_2 \dot{\delta}_1 + \frac{3}{2} \delta_2 \dot{\delta}_2\right) \quad \Longrightarrow \quad K \begin{bmatrix} \frac{5}{2}K & -\frac{1}{2}K \\ -\frac{1}{2}K & \frac{3}{2}K \end{bmatrix}$$